Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Sound and Vibration 271 (2004) 535–545
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Non-linear vibration of composite beams with an arbitrary delamination ☆

Song-Nan Luo[a,b,*], Fu Yi-Ming[a], Cao Zhi-Yuan[b]

[a] *Department of Engineering Mechanics, Hunan University, Changsha 410082, China*
[b] *Department of Engineering Mechanics and Technology, Tongji University, Shanghai 200092, China*

Received 16 August 2000; accepted 10 March 2003

Abstract

In this paper, the non-linear vibration, including the transverse shear, is investigated for composite beams with an arbitrary delamination through the width. The effects of different positions and sizes of the delamination on non-linear vibration of beams are considered. The amplitude–frequency curves of non-linear free vibration are obtained.

1. Introduction

The composite materials, due to their specific natures, have been applied widely in the engineering. However, the static and the dynamic features of the composite constructions will be affected significantly by the delaminations that occur in the imperfect manufacturing or in loading. Therefore, the static and dynamics studies for delaminated composite constructions are very important.

Non-linear dynamic analysis of beams have been of considerable research interest in the recent years. Xia et al. [1] analyzed the harmonic responses of beams with longitudinal and transversal coupling by the incremental harmonic balance method. Kar and Dwiredy [2,3] investigated the non-linear dynamic behavior of a slender beam carrying a lumped mass subjected to principal parametric base excitation. The vibration of a split beam was researched by Wang et al. [4]. They observed that the fundamental frequency was not visibly reduced due to the short delamination. Later, the vibrations of a symmetric delaminated beam plate relative to buckled were researched by Yin and Jane [5]. They obtained that some new vibration modes and frequencies depend

☆ This work is supported by National Natural Science Foundation of China.
*Corresponding author.
E-mail address: luosnl@yahoo.com.cn (S.-N. Luo).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00279-7

sensitively on the delamination length, the location and on the magnitude of the post-buckling load. Moreover, Chane and Liane [6] studied the free vibrations of delaminated beam plates with respect to post-buckling referential states.

The study about non-linear vibration of composite beams with arbitrary delamination is scarce. In this paper, the beams with an arbitrary delamination through the width are divided into four regions. The basic equations are built in each region and the continuous conditions are founded. The B-specimen functions are used to describe the variations in the space, and the dimensionless differential equations about time are obtained by using Galerkin's method. Finally, the amplitude–frequency curves of the non-linear vibration of composite beams with an arbitrary delamination are obtained by using the incremental harmonic balance method [7].

2. Basic equations

Consider a composite beam with an arbitrary delamination under the axial force N^0 and the transverse distributed force p and the beam is divided into four regions, respectively denoted I–IV as shown in Fig. 1. Supposing the thickness h in regions I and IV h_2 in regions II, h_3 in regions III, also $h_2 + h_3 = h$, the width is one unit, The distances from the middle surface of each region to the top or bottom surface of the beam are, respectively, denoted by t_1^i and t_2^i and the upper marks $i =$ I–IV.

Now, consider a general beam, the displacement components u and w of any point that include the effect of transverse shear deformation may be described as follows:

$$u(x,z,t) = u^0(x,t) - z\phi(x,t), \quad w(x,z,t) = w^0(x,t), \tag{1}$$

where u^0 and w^0 are the values of u and w at the middle surface and ϕ is the rotation angle of the normal to the middle surface in the xz-plane. Taking the Green non-linear strain–displacement relations, we obtain

$$\varepsilon_x = u^0_{,x} - z\phi_{,x} + \tfrac{1}{2}w^2_{,x}, \quad \gamma_{xz} = w_{,x} - \phi. \tag{2}$$

Because of $\sigma_y = \tau_{yz} = \tau_{xy} = 0$, the stress–strain relation for the kth layer can be written as follows:

$$\begin{Bmatrix} \sigma_x^{(k)} \\ \sigma_{zx}^{(k)} \end{Bmatrix} = \begin{bmatrix} C_{11}^{(k)} & C_{15}^{(k)} \\ C_{15}^{(k)} & C_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{zx} \end{Bmatrix}, \tag{3}$$

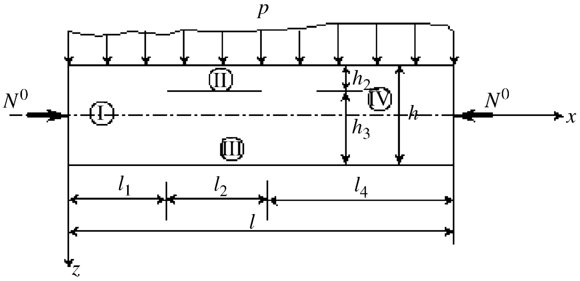

Fig. 1. Configuration of beam with an arbitrary delamination.

in which $\sigma_{ij}^{(k)}$ and ε_{ij} are the Kirchoff stress components and the Lagrange strain components, respectively, and $C_{ij}^{(k)}$ are elastic stiffnesses in the kth layer, The membrane stress resultants N, the shear force Q and the stress couples M are obtained as follows:

$$\begin{aligned} -N &= A_1 u^0_{,x} + \tfrac{1}{2}A_1 w^2_{,x} - B_1\phi_{,x} + A_2 w_{,x} - A_2\phi, \\ Q &= A_2 u^0_{,x} + \tfrac{1}{2}A_2 w^2_{,x} - B_2\phi_{,x} + A_3 w_{,x} - A_3\phi, \\ M &= B_1 u^0_{,x} + \tfrac{1}{2}B_1 w^2_{,x} + D_1\phi_{,x} + B_2 w_{,x} - B_2\phi, \end{aligned} \tag{4}$$

where A_1, A_2, A_3, B_1, B_2 and D_1 are the integral constants related to material and geometrical parameters of the transverse section, and

$$\begin{aligned} A_1 &= \textstyle\int_{-t_2}^{t_1} c_{11}^{(k)}\,\mathrm{d}z, \quad A_2 = \int_{-t_2}^{t_1} c_{15}^{(k)}\,\mathrm{d}z, \quad A_3 = \int_{-t_2}^{t_1} c_{55}^{(k)}\,\mathrm{d}z, \\ B_1 &= \textstyle\int_{-t_2}^{t_1} z c_{11}^{(k)}\,\mathrm{d}z, \quad B_2 = \int_{-t_2}^{t_1} z c_{15}^{(k)}\,\mathrm{d}z, \quad D_1 = \int_{-t_2}^{t_1} z^2 c_{11}^{(k)}\,\mathrm{d}z, \end{aligned}$$

where $A_2 = B_2 = 0$, if we only consider the special orthotropic composite ($C_{15}^{(k)} = 0$) beam and $B_1 = 0$, if the x-axis is set at neutral layer of the beam.

Neglecting the influence of axial inertia and rotary inertia and considering the equilibrium in x and z directions, the non-linear equations of motion of the beam can be written

$$\begin{aligned} & N_{,x} + \bar{\gamma}(Q\phi)_{,x} = 0, \\ & Q_{,x} - (N w_{,x})_{,x} + p - \rho A\ddot{w} = 0, \\ & Q - M_{,x} = 0, \end{aligned} \tag{5}$$

where $\bar{\gamma}$ is a factor of shear force representing the influence of shear force on the axial force [8]. The influence is considered when $\bar{\gamma} = 1$, and the influence is neglected when $\bar{\gamma} = 0$. The influence always is considered in present analysis, so, $\bar{\gamma}$ always equals 1.

Eliminating u^0 from Eq. (4), substituting the expressions of Q and M into Eq. (5) and only considering the special orthotropic composite beam, we obtain

$$\begin{aligned} & N_{,x} + A_3(w_{,xx} - \phi_{,x})\phi + A_3(w_{,x} - \phi)\phi_{,x} = 0, \\ & A_3(w_{,xx} - \phi_{,x}) - N w_{,xx} - N_{,x} w_{,x} + p - \rho A\ddot{w} = 0, \\ & D_1\phi_{,xx} + A_3(w_{,x} - \phi) = 0. \end{aligned} \tag{6}$$

Introducing the dimensionless parameters as follows:

$$\begin{aligned} & \xi^i = \frac{x^i}{l^i}, \quad \zeta^i = \frac{z^i}{h}, \quad W^i = \frac{w^i}{h}, \quad \Phi^i = \phi^i, \quad \alpha_1^i = \frac{t_1^i}{h}, \quad \alpha_2^i = \frac{t_2^i}{h}, \\ & \beta^i = \frac{h}{l^i}, \quad k^i = \frac{l^i}{l}, \quad \tau = \frac{t}{l}\sqrt{\frac{A_1^{\mathrm{I}}}{\rho\delta h}}, \quad P^i = \frac{p^i l^i}{A_1^{\mathrm{I}}}, \\ & \bar{C}^i = \frac{A_3}{A_1^{\mathrm{I}}}, \quad \bar{D}^i = \frac{D_1^i}{A_1^{\mathrm{I}} h^2} \quad \bar{N}^i = \frac{N^i}{A_1^{\mathrm{I}}}, \quad \bar{Q}^i = \frac{Q^i}{A_1^{\mathrm{I}}}, \quad \bar{M}^i = \frac{M^i}{A_1^{\mathrm{I}} h}, \end{aligned} \tag{7}$$

where x^i, z^i, ξ^i and ζ^i are the local co-ordination, substituting Eq. (7) into Eq. (6) and considering each region for the delaminated beam, respectively, we obtain the dimensionless governing equations for the delaminated beam

$$
\begin{aligned}
&\bar{N}^i_{1\xi} + \bar{C}^i(\beta^i W^i_{,\xi\xi} - \Phi^i_{,\xi})\Phi^i + \bar{C}^i(\beta^i W^i_{,\xi} - \Phi^i)\Phi^i_{,\xi} = 0,\\
&\bar{C}^i(\beta^i W^i_{\xi\xi} - \Phi^i_{,\xi}) - \beta^i \bar{N}^i W^i_{,\xi\xi} - \beta^i \bar{N}^i_{,\xi} W^i_{,\xi} + P^i - \beta^i k^{i2} \ddot{W}^i = 0,\\
&\bar{D}^i \beta^{i2} \Phi^i_{,\xi\xi} + \bar{C}^i(\beta^i W^i_{,\xi} - \Phi^i) = 0.
\end{aligned} \tag{8}
$$

The dimensionless expressions of internal forces are

$$\bar{M}^i = \bar{D}^i \beta^i \Phi^i_{,\xi}, \quad \bar{Q}^i = \bar{C}^i(\beta^i \bar{W}^i_{,\xi} - \Phi^i). \tag{9}$$

The dimensionless boundary conditions are

$$
\begin{aligned}
&\text{at } \xi^{\mathrm{I}} = 0, \bar{N}^{\mathrm{I}}, W^{\mathrm{I}} \quad \text{or} \quad \bar{Q}^{\mathrm{I}} + \bar{N}^{\mathrm{I}} W^{\mathrm{I}}_{,\xi}\beta^{\mathrm{I}}, \Phi^{\mathrm{I}} \quad \text{or} \quad \bar{M}^{\mathrm{I}} \text{ are given}\\
&\text{at } \xi^{\mathrm{IV}} = 1, \bar{N}^{\mathrm{IV}}, W^{\mathrm{IV}} \quad \text{or} \quad \bar{Q}^{\mathrm{IV}} + \bar{N}^{\mathrm{IV}} W^{\mathrm{IV}}_{,\xi}\beta^{\mathrm{IV}}, \Phi^{\mathrm{IV}} \quad \text{or} \quad \bar{M}^{\mathrm{IV}} \text{ are given.}
\end{aligned} \tag{10}
$$

The dimensionless continuity conditions for the displacements are
At the left end of the delamination:

$$
\begin{aligned}
&W^{\mathrm{I}} = W^{\mathrm{II}}, \quad \Phi^{\mathrm{I}} = \Phi^{\mathrm{II}},\\
&W^{\mathrm{I}} = W^{\mathrm{III}}, \quad \Phi^{\mathrm{I}} = \Phi^{\mathrm{III}}.
\end{aligned}
$$

At the right end of the delamination:

$$
\begin{aligned}
&W^{\mathrm{IV}} = W^{\mathrm{II}}, \quad \Phi^{\mathrm{IV}} = \Phi^{\mathrm{II}},\\
&W^{\mathrm{IV}} = W^{\mathrm{III}}, \quad \Phi^{\mathrm{IV}} = \Phi^{\mathrm{III}}.
\end{aligned} \tag{11}
$$

The dimensionless equilibrium conditions for the internal forces are
At the left section of the delamination:

$$
\begin{aligned}
&\bar{M}^{\mathrm{I}} = \bar{M}^{\mathrm{II}} + \bar{M}^{\mathrm{III}}, \quad \bar{Q}^{\mathrm{I}} = \bar{Q}^{\mathrm{II}} + \bar{Q}^{\mathrm{III}},\\
&\bar{N}^{\mathrm{I}} = \bar{N}^{\mathrm{II}} + \bar{N}^{\mathrm{III}}.
\end{aligned}
$$

At the right section of the delamination:

$$
\begin{aligned}
&\bar{M}^{\mathrm{IV}} = \bar{M}^{\mathrm{II}} + \bar{M}^{\mathrm{III}}, \quad \bar{Q}^{\mathrm{IV}} = \bar{Q}^{\mathrm{II}} + \bar{Q}^{\mathrm{III}},\\
&\bar{N}^{\mathrm{IV}} = \bar{N}^{\mathrm{II}} + \bar{N}^{\mathrm{III}}.
\end{aligned} \tag{12}
$$

3. Method of solution

As in Refs. [8,11], considering the additional axial force influenced by non-linear, we separate the axial force into two terms. The first term describes the axial force applied at the two ends, the second term describes the varying axial force with dimensionless co-ordinate ξ and the force being influenced by geometrical non-linear deformation. So, as usual a solution of Eq. (8) is sought in the separable form

$$\bar{N}^i = \gamma^i \bar{N}^0(\tau) + \sum_{m=-1}^{S^i+1} N_m^i(\tau) F_m^i(\xi),$$
$$W^i = \sum_{m=-1}^{S^i+1} W_m^i(\tau) G_m^i(\xi),$$
$$\Phi^i = \sum_{m=-1}^{S^i+1} \Phi_m^i(\tau) G_m^{i'}(\xi), \tag{13}$$

in which, $\gamma^i = 1$when $i = \mathrm{I},\ \mathrm{IV}$; $\gamma^{\mathrm{II}} + \gamma^{\mathrm{III}} = 1$ when $i = \mathrm{II}, \mathrm{III}$ and their values are distributed according to the ratio of A_1^{II} and A_1^{III}. The N_m^i, W_m^i and Φ_m^i are the functions of the time τ. $F_m^i(\xi)$ and $G_m^i(\xi)$ are the basic functions of ξ that, respectively, relates to cubic B-specimen functions $\Omega_3(\xi)$ and five order B-specimen functions $\Omega_5(\xi)$. And suppose that the each region is divided evenly by $\Delta^i = 1/S^i$, s^i is the number of specimen points in the region i.

The general expressions of $F_m^i(\xi)$ are

$$F_m^i(\xi) = \Omega_3\left(\frac{\xi}{\Delta^i} - m\right).$$

Note that $N^0(\tau)$ has satisfied the boundary conditions with zero when let $\gamma^i = 1$ in Eq. (13), in order to satisfy the boundary conditions for second term of axial force equating zero at both ends in Eq. (13) the some of the expressions must be changed as follows:

$$F_0^{\mathrm{I}}(\xi) = \Omega_3\left(\frac{\xi}{\Delta^{\mathrm{I}}}\right) - 4\Omega_3\left(\frac{\xi}{\Delta^{\mathrm{I}}} - 1\right),$$
$$F_1^{\mathrm{I}}(\xi) = \Omega_3\left(\frac{\xi}{\Delta^{\mathrm{I}}}\right) - 4\Omega_3\left(\frac{\xi}{\Delta^{\mathrm{I}}} + 1\right),$$
$$F_{s^{\mathrm{IV}}-1}^{\mathrm{IV}}(\xi) = \Omega_3\left(\frac{\xi}{\Delta^{\mathrm{IV}}} - s^{\mathrm{IV}}\right) - 4\Omega_3\left(\frac{\xi}{\Delta^{\mathrm{IV}}} - s^{\mathrm{IV}} + 1\right),$$
$$F_{s^{\mathrm{IV}}}^{\mathrm{IV}}(\xi) = \Omega_3\left(\frac{\xi}{\Delta^{\mathrm{IV}}} - s^{\mathrm{IV}}\right) - 4\Omega_3\left(\frac{\xi}{\Delta^{\mathrm{IV}}} - s^{\mathrm{IV}} - 1\right),$$

in which, the expression of $\Omega_3(x)$ is

$$\Omega_3(x) = \frac{1}{6}\begin{cases} (x+2)^3, & x \in [-2, -1], \\ (x+2)^3 - 4(x+1)^3, & x \in [-1, 0], \\ (2-x)^3 - 4(1-x)^3, & x \in [0, 1], \\ (2-x)^3, & x \in [1, 2], \\ 0, & |x| \geqslant 2. \end{cases}$$

The general expressions of $G_m^i(\xi)$ are

$$G_m^i(\xi) = \Omega_5\left(\frac{\xi}{\Delta^i} - m\right).$$

However, some of the expressions must be changed as follows for satisfying the simple supported boundary conditions at both ends

$$G_0^I(\xi) = \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}\right) - 3\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}+1\right) + 12\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}+2\right),$$

$$G_1^{\mathrm{I}}(\xi) = \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}-1\right) - \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}+1\right),$$

$$G_2^{\mathrm{I}}(\xi) = \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}\right) - 3\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}-1\right) + 12\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}-2\right),$$

$$G_{s^{\mathrm{IV}}-2}^{\mathrm{IV}}(\xi) = \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}\right) - 3\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}+1\right) + 12\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}+2\right),$$

$$G_{s^{\mathrm{IV}}-1}^{\mathrm{IV}}(\xi) = -\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}-1\right) + \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}+1\right),$$

$$G_{s^{\mathrm{IV}}}^{\mathrm{IV}}(\xi) = \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}\right) - 3\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}-1\right) + 12\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}-2\right)$$

and for the clamped boundary conditions at both ends

$$G_0^{\mathrm{I}}(\xi) = \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}+1\right) - \frac{16}{66}\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}\right) - 10\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}+2\right),$$

$$G_1^{\mathrm{I}}(\xi) = -\frac{26}{33}\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}\right) + \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}-1\right) + \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}+1\right),$$

$$G_2^{\mathrm{I}}(\xi) = -10\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}-2\right) + \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}-1\right) - \frac{16}{66}\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{I}}}\right),$$

$$G_{s^{\mathrm{IV}}-2}^{\mathrm{IV}}(\xi) = -10\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}+2\right) + \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}+1\right) - \frac{16}{66}\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}\right),$$

$$G_{s^{\mathrm{IV}}-1}^{\mathrm{IV}}(\xi) = -\frac{26}{33}\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}\right) + \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}-1\right) + \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}+1\right),$$

$$G_{s^{\mathrm{IV}}}^{\mathrm{IV}}(\xi) = \Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}-1\right) - \frac{16}{66}\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}\right) - 10\Omega_5\left(\frac{\xi}{\Delta^{\mathrm{IV}}}-s^{\mathrm{IV}}-2\right),$$

in which, the expression of $\Omega_5(x)$ is

$$\Omega_5(x) = \frac{1}{120}\begin{cases} (x+3)^5, & x\in[-3,-2], \\ (x+3)^5 - 6(x+2)^5, & x\in[-2,-1], \\ (x+3)^5 - 6(x+2)^5 - 15(x+1)^5, & x\in[-1,0], \\ (3-x)^5 - 6(2-x)^5 + 15(1-x)^5, & x\in[0,1], \\ (3-x)^5 - 6(2-x)^5, & x\in[1,2], \\ (3-x)^5, & x\in[2,3], \\ 0, & |x|\geqslant 3. \end{cases}$$

According to the above expressions, the F_m^i and G_m^i can satisfy the displacement conditions at both ends when $m \neq -1$ and $s^\imath + 1$.

Substituting Eq. (13) into Eq. (8), performing the Galerkin's integrations in each region, resolving out the N_m^i from the first resulting equation, then substituting N_m^i into the latter two resulting equations we obtain

$$\begin{aligned}&(a^i_{4mj} + \gamma^i \bar{N}^0 a^i_{13mj})W^i_m + a^i_{5mj}\Phi^i_m + a^i_{16kmnj}W^i_k\Phi^i_m W^i_n\\&\quad + a^i_{17kmnj}\Phi^i_k\Phi^i_m W^i_n + a^i_{7j}P^i + a^i_{8mj}\ddot{W}^i_m = 0,\\&a^i_{18mnj}W^i_m\Phi^i_n + a^i_{10mj}\Phi^i_m + a^i_{11mj}W^i_m + a^i_{19mnj}\Phi^i_m\Phi^i_n = 0,\end{aligned} \tag{14}$$

where $a_4^i, \ldots, a_{19}^i$ are the integral constant that relate to the $F_m^i(\xi)$ and $G_m^i(\xi)$.

Eq. (14) are the basic governing equations for solving the non-linear vibrations of the delaminated beams. In these equations, the coupling of longitudinal and transversal motions and the effect of transverse shear deformation are included. Due to the number of unknown quantities is reduced, so that, the solving of the basic equations becomes easy.

Substituting Eq. (13) into the expressions of internal forces, boundary conditions and continuous conditions, also the corresponding expressions can be obtained.

The external loads on the beam are supposed in the following forms:

$$\begin{aligned}&P^i(\tau) = P^i_0 + P^i_\tau \cos\theta\tau,\\&\bar{\mathrm{N}}^0(\tau) = \mathrm{N}_0 + N_\tau \cos\theta\tau.\end{aligned} \tag{15}$$

Using the incremental harmonic method and setting

$$\begin{aligned}&N_\tau = N_{0\tau} + \Delta N_\tau, \quad P^i_\tau = P^i_{0\tau} + \Delta P^\imath_\tau, \quad \theta = \theta_0 + \Delta\theta,\\&W^i_m = W^i_{0m} + \Delta W^i_m, \quad \Phi^i_m = \Phi^i_{0m} + \Delta\Phi^i_m\end{aligned} \tag{16}$$

letting $\bar{t} = \theta\tau$, $\bar{t}_0 = \theta_0\tau$, $\Delta\bar{t} = \Delta\theta\tau$, and substituting Eqs. (15) and (16) into Eq. (14), we obtain:

$$\begin{aligned}&a^i_{8mj}\theta_0^2\Delta\ddot{W}^i_m + (b^i_{1mj} + b^i_{2mj}\cos\bar{t})\Delta W^i_m + b^i_{3mj}\Delta\Phi^i_m\\&\quad = r^i_{1j} - 2\theta_0 L^i_{1j}\Delta\theta - L^i_{2j}\Delta P^i_\tau - L^i_{3j}\Delta N_\tau,\\&b^i_{4mj}\Delta W^i_m + b^i_{5mj}\Delta\Phi^i_m = r^i_{2j},\end{aligned} \tag{17}$$

where $b^i_{1mj}, b^i_{5mj}, L^i_{1j}, \ldots, L^i_{3j}, \ldots$ are the constants that related to $a^i_1, \ldots, a^i_{19}$, W^i_{0m} and Φ^i_{0m}. The residuals r^i_{1j} and r^i_{2j} can be written as follows:

$$\begin{aligned}r^i_{1j} = &-(a^i_{4mj} + \gamma^i N_0 a^i_{13mj} + \gamma^i N_{0\tau} a^i_{13mj}\cos\bar{t})W^i_{0m}\\&- a^i_{5mj}\Phi^i_{0m} - a^i_{16kmnj}W^i_{0k}\Phi^i_{0m}W^i_{0n}\\&- a^i_{17kmnj}\Phi^i_{0k}\Phi^i_{0m}W^i_{0n} - a^i_{7j}P^i_0 - a^i_{7j}P^i_{0\tau}\cos\bar{t} - a^i_{8mj}\theta_0^2\ddot{W}^i_{0m},\\r^i_{2j} = &-a^i_{18mnj}W^i_{0m}\Phi^i_{0n} - a^i_{10mj}\Phi^i_{0m} - a^i_{11mj}W^i_{0m} - a^i_{19mnj}\Phi^i_{0m}\Phi^i_{0n}.\end{aligned} \tag{18}$$

Eq. (17) can be solved for the unknown functions ΔW^i_m, $\Delta\Phi^i_m$ and $\Delta\theta$ when a set of W^i_{0m}, Φ^i_{0m} and N^0 is given in which the corresponding increment is set to zero at each incremental step. By carrying out the incremental computation procedure as presented in Ref. [8], the exciting frequency θ, the transverse deflection W^i_m and the rotation angle Φ^i_m can be determined.

4. Numerical results and discussion

Numerical results for non-linear free vibration of composite beams with an arbitrary delamination are presented. Set

$$N_{0\tau} = \Delta N_\tau = 0, \quad P_0^i = P_{0\tau}^i = \Delta P_\tau^i = 0$$

and expand the unknowns W_{0m}^i, Φ_{0m}^i, ΔW_m^i and $\Delta\Phi_m^i$ into Fourier series in $\bar{t}$:

$$\begin{aligned}
W_{0m}^i(\bar{t}) &= \sum_{k=0,1,2,\ldots}^{\infty} \left(A_{km}^i \sin\frac{k\bar{t}}{2} + B_{km}^i \cos\frac{k\bar{t}}{2} \right), \\
\Phi_{0m}^i(\bar{t}) &= \sum_{k=0,1,2,\ldots}^{\infty} \left(C_{km}^i \sin\frac{k\bar{t}}{2} + D_{km}^i \cos\frac{k\bar{t}}{2} \right), \\
\Delta W_m^i(\bar{t}) &= \sum_{k=0,1,2,\ldots}^{\infty} \left(\Delta A_{km}^i \sin\frac{k\bar{t}}{2} + \Delta B_{km}^i \cos\frac{k\bar{t}}{2} \right), \\
\Delta\Phi_m^i(\bar{t}) &= \sum_{k=0,1,2,\ldots}^{\infty} \left(\Delta C_{km}^i \sin\frac{k\bar{t}}{2} + \Delta D_{km}^i \cos\frac{k\bar{t}}{2} \right).
\end{aligned} \tag{19}$$

Substituting Eq. (19) into Eq. (17), and equating the coefficients of $\sin(k\bar{t}/2)$ and $\cos(k\bar{t}/2)$ terms, a set of linear algebraic equation can be obtained as follows:

$$\begin{aligned}
b_{6mj}^i \Delta W_m^i + b_{3mj}^i \Delta\Phi_m^i &= r_{1j}^i - 2\theta_0 L_{1j}^i \Delta\theta, \\
b_{4mj}^i \Delta W_m^i + b_{5mj}^i \Delta\Phi_m^i &= r_{2j},
\end{aligned} \tag{20}$$

where $b_{6mj}^i = a_{8mj}^i((k/2)\theta_0)^2 + b_{1mj}^i$.

In the solving processes, only the lowest vibration model of the beam is considered. θ_1 represents the dimensionless foundational frequency. And take $k = 0, 1, 2, 3, 4$ for Eq. (19) in the calculation.

According to the above methods, vibration analysis (Ref. [9]) and buckling analysis (Ref. [12]) of composite beams with arbitrary delamination have been given. In Ref. [9], the dimensionless foundational frequencies of beams without delamination are calculated by using present method and are compared with by theory and the maximum error is 0.83%. In Ref. [12], the dimensionless buckling loads of composite beams with delamination are calculated according to the present method and are compared with the results in Ref. [13], and the two results are identical. These show that present methods are correct.

In Figs. 2–6, the amplitude–frequency curves of non-linear free vibration of the composite beam with clamping of two ends are plotted. Suppose the number of layers is five $0°/90°/0°/90°/0°$, $l = 1$ m, $l/h = 30$, $\bar{l}_1 = l_1/l$ that represents the delaminated position in x direction, $\bar{h}_2 = h_2/h$ that represents the delaminated position in z direction, $\bar{k} = l_2/l$ that represents the delaminated length. And θ_1 is the fundamental linear foundational frequency and $W_{\max}$ is the maximum displacement in z direction. Also the material elastic constants of the beam in each layer [10] are: $E_1 = 172.4$ GPa, $E_2 = 7.79$ GPa, $G_{12} = 5.3$ GPa, $v_{12} = 0.21$.

Figs. 2 and 3 show that the frequency are larger with larger amplitude in non-linear free vibration. A harder non-linear character is shown in the amplitude–frequency response curves. As

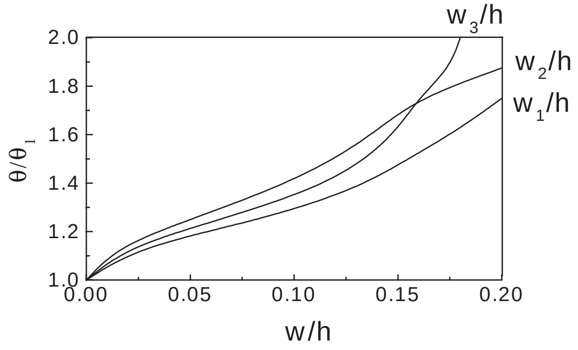

Fig. 2. Amplitude–frequency response curves in each region: ($\bar{N}^0 = 0.8\bar{N}_{cr}$, $\bar{k} = 0.1$, $\bar{l}_1 = 0.45$, $\bar{h}_2 = 0.2$).

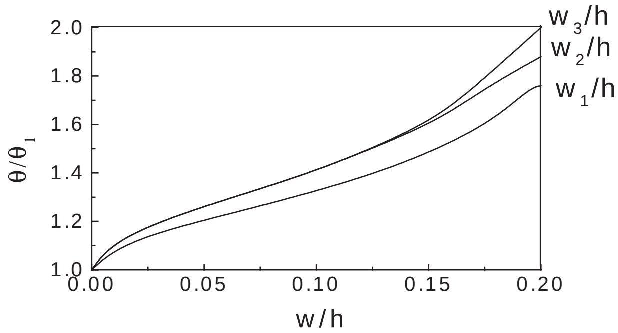

Fig. 3. Amplitude–frequency response curves in each region: ($\bar{N}^0 = 0.8\bar{N}_{cr}$, $\bar{k} = 0.1$, $\bar{l}_1 = 0.45$, $\bar{h}_2 = 0.4$).

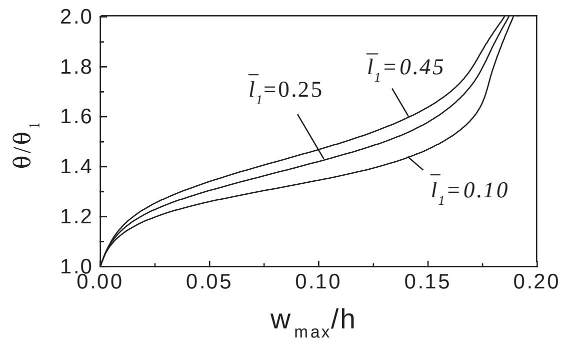

Fig. 4. Effects of the different positions of delamination on the amplitude–frequency response curves: ($\bar{N}^0 = 0.8\bar{N}_{cr}$, $\bar{k} = 0.1$, $\bar{h}_2 = 0.2$).

the amplitude is small, the frequency increases greatly with the amplitude. However, as the amplitude is larger, the frequency increases slowly with the amplitude. As the amplitude is further larger, the frequency also increases greatly with the amplitude.

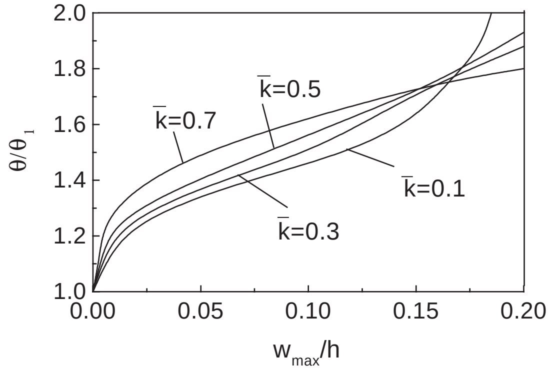

Fig. 5. Effects of the different length of delamination on the amplitude–frequency response curves: ($\bar{N}^0 = 0.8\bar{N}_{cr}$, $\bar{l}_1 = 1/2(1-\bar{k})$, $\bar{h}_2 = 0.2$).

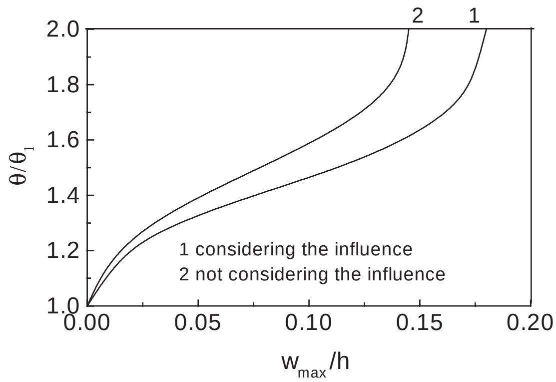

Fig. 6. Effects of the transverse shear deformation on the amplitude–frequency response curves: ($\bar{N}^0 = 0.8\bar{N}_{cr}$, $\bar{k} = 0.1$, $\bar{h}_2 = 0.2$, $\bar{l}_1 = 0.45$).

The amplitude–frequency response curves are compared for different positions of delamination in Fig. 4. The effects of the different positions of delamination on the amplitude–frequency response curves are small and this character is similar to that in Ref. [9]. As the amplitude is larger, the effects become obvious. But as the amplitude is further larger, the effects become weaken.

The amplitude–frequency response curves are compared for different length of delamination in Fig. 5. As the amplitude is small, the frequency is larger with the length of delamination being longer. As the amplitude is larger and the length of delamination increases, the non-linear character becomes weak, i.e., hard shape from strong hard shape. So the frequency increases slowly with the longer length of delamination.

The effects of the transverse shear deformation on the amplitude–frequency response curves is shown in Fig. 6. It indicates that the influence of transverse shear deformation cannot be ignored for non-linear vibration of composite beam.

A comparison of the amplitude–frequency response curves for different materials is shown in Fig. 7. The other two materials are boron-epoxy composite material [10] and glass-epoxy

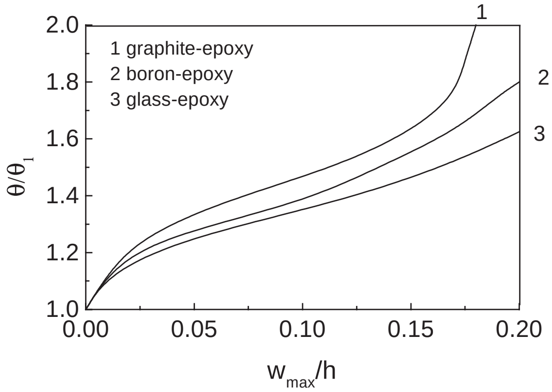

Fig. 7. Effects of amplitude–frequency response curves on the transverse shear deformation: ($\bar{N}^0 = 0.8\bar{N}_{cr}$, $\bar{k} = 0.1$, $\bar{h}_2 = 0.2$, $\bar{l} = 0.45$).

composite material [11] and the material constants are: $E_1 = 137.9\,\text{GPa}$, $E_2 = 14.48\,\text{GPa}$, $G_{12} = 5.86\,\text{GPa}$, $v_{21} = 0.21$ and $E_1 = 53.8\,\text{GPa}$, $E_2 = 17.93\,\text{GPa}$, $G_{12} = 8.96\,\text{GPa}, v_{21} = 0.25$. As the amplitude is small, the effects of the different material on the amplitude–frequency response cures are small. As the amplitude is larger, the hard characters of the non-linear vibration become obvious with the larger ratio of E_1/E_2.

References

[1] P.Q. Xia, Dominic Pun harmonic responses of beams with longitudinal and transversal coupling, Journal of Vibration Engineering (Chinese) 8 (1) (1995) 67–72.

[2] R.C. Kar, S.K. Dwiredy, Non-linear dynamics of a slender beam carrying a lumped mass with principal parametric and internal resonances, International Journal of Non-Linear Mechanics 34 (3) (1999) 515–529.

[3] S.K. Dwiredy, R.C. Kar, Dynamics a slender beam with an attacked mass under combination parametric and internal resonance. Part II: periodic chaotic responses, Journal of Sound and Vibration 222 (2) (1999) 281–305.

[4] J.T.S. Wang, Y.Y. Liu, J.A. Gibby, Vibration of split beam, Journal of Sound Vibration 84 (1982) 491–502.

[5] W.L. Yin, K.C. Jane, Vibration of a delaminated beam-plate relative to buckled states, Journal of Sound and Vibration 156 (1992) 125–140.

[6] T.P. Chang, J.Y. Liang, Vibration of post-buckled delaminated beam-plate, International Journal of Solids Structures 35 (12) (1998) 1199–1217.

[7] S.L. Lau, W.S. Zhang, Nonlinear vibration of piecewise-linear systems by incremental harmonic balance method, Journal of Applied Mechanics 59 (1992) 153–160.

[8] S.N. Luo, Y.M. Fu, Z.Y. Cao, Interaction of shearing and axial forces in nonlinear dynamic analysis of beam, Journal of Hunan University (Natural Sciences Edition) (Chinese) 26 (9) (1999) 12–42.

[9] S.N. Luo, Y.M. Fu, Z.Y. Cao, Vibration analysis of composite beams with arbitrary delamination, Structure and Envirorment Engineering (Chinese) 1 (2000) 80–84.

[10] Andrzej Tylilcowski. Dynamic stability of nonlinear antisymmetrically laminated cross-ply rectangular plates, Journal of Applied Mechanics 56 (1989) 375–381.

[11] C.Y. Chia, Nonlinear Analysis of Plates, McGraw-Hill, New York, 1980.

[12] S.N. Luo, Z.Y. Cao, Y.M. Fu, Buckling analysis of composite beams with delamination, Chinese Quarterly of Mechanics (Chinese) 22 (3) (2001) 322–328.

[13] H.Y. Huang, G.A. Kardomates, Buckling of orthotropic beam-plates with multiple delamination, International Journal of Solid Structures 35 (13) (1998) 1355–1362.